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LIOUVILLE THEOREMS, PARTIAL REGULARITY AND HÖLDER CONTINUITY OF WEAK SOLUTIONS TO QUASILINEAR ELLIPTIC SYSTEMS

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ABSTRACT. This paper describes the connections between Liouville type theorems and interior regularity results for bounded weak solutions of quasilinear elliptic systems with quadratic growth. It is shown that equivalence does in general hold only in some restricted sense. A complete correspondence can be established in certain cases, e.g. for small solutions and for minima of quadratic variational integrals.

0. Introduction. Several years ago, Frehse [3, 2] pointed out that "the experience in the theory of elliptic equations suggests that regularity theorems hold for those types of equations for which a Liouville theorem is true". This conjecture was later on verified by Giaquinta-Nečas [8] and by Kawohl [14], in the case of elliptic systems with right-hand side depending linearly upon the derivatives,

$$(0.1)\qquad -D_\alpha\left(a_{ik}^{\alpha\beta}(x,u)D_\beta u^k\right) = c_{ik}^{\beta}(x,u)D_\beta u^k \qquad (i=1,\dots,N).$$

More precisely, Kawohl [14] proved that every bounded weak solution u of (0.1) in some open domain $\Omega \subset \mathbf{R}^n$ is Hölder continuous if and only if the principal part of the system has the "Liouville property"; i.e. if for every $x_0 \in \Omega$ any bounded entire solution $u_0 \in H^1_{2,\mathrm{loc}} \cap L_\infty(\mathbf{R}^n, \mathbf{R}^N)$ of the system

$$-D_\alpha\left(a_{ik}^{\alpha\beta}(x_0,u_0)D_\beta u_0^k\right) = 0 \qquad (i=1,\dots,N)$$

has to be a constant vector. In addition, it was shown that the Liouville property implies local a priori bounds for the Hölder norms of the bounded weak solutions of (0.1).

Our paper deals with the corresponding problem for quasilinear elliptic systems of the form

$$(0.2)\qquad -D_\alpha\left(a_{ik}^{\alpha\beta}(x,u)D_\beta u^k\right) = f^i(x,u,\nabla u) \qquad (i=1,\dots,N),$$

the right-hand side of which grows quadratically with respect to ∇u. In particular, we shall treat the case when $f^i(x,u,\nabla u) = b_{ikl}^{\alpha\beta}(x,u)D_\alpha u^k D_\beta u^l$, $b_{ikl}^{\alpha\beta}$ and $a_{ik}^{\alpha\beta}$ being

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bounded continuous coefficients. It is worth noting that this structure includes the Euler equations of quadratic multiple integrals

$$\int_\Omega a_{ik}^{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^k \, dx. \tag{0.3}$$

As has already been observed by various authors, Liouville type theorems and regularity results for such systems can often be obtained under essentially the same assumptions and by similar methods of proof (cf. Hildebrandt and Widman **[11]** and Meier **[16, 17]**).[1] Some of these connections are explained by a remark due to Ivert **[12]** who showed, by using a blow-down argument, that certain a priori bounds for the modulus of continuity of bounded weak solutions of (0.2) imply Liouville theorems.

It turns out, however, that there are even two-dimensional systems of the form (0.2) for which a Liouville theorem holds and which also possess discontinuous bounded solutions. An example is given in §2 below.

On the other hand, we prove that a Liouville property for (0.2) implies the Hölder continuity of those bounded weak solutions which belong to the Sobolev class $H^1_{2+\varepsilon,\mathrm{loc}}$ for some $\varepsilon > 0$ and whose gradients satisfy a reverse Hölder inequality (Theorem 3). Furthermore, if there exists a nonconstant, bounded entire solution satisfying a reverse Hölder inequality at infinity, we shall construct also a discontinuous solution of the same system (Theorem 4). As a corollary, one obtains the complete correspondence between Liouville theorems and regularity theorems in the cases where partial regularity results for (0.2) have been established by Giaquinta and Giusti (cf. **[4, 5]**).

The proofs of the theorems rest upon a blow-up and blow-down argument similar to **[8, 14]**. In order to carry over the method to systems (0.2) with quadratically growing nonlinearity, one must, in addition, verify the *strong* H^1_2-convergence of a weakly convergent sequence of blown-up functions. A main purpose of our paper is to demonstrate how this improved convergence property can be deduced from local $H^1_{2+\varepsilon}$-bounds for the solutions of (0.2).

The results concerning quasilinear systems are presented in §4, while §3 deals with the equivalence of Liouville theorems and regularity theorems for bounded local minima of (nondifferentiable) quadratic functionals (0.3). Here we make use of various estimates which are essentially contained in recent papers by Giaquinta and Giusti (cf. **[6, 7]**). The problem of showing strong H^1_2-convergence does not arise in this context, since one can apply the same weak lower semicontinuity argument as in **[7]**. On the other hand, §3 already displays some basic aspects and techniques that will be important in §4.

1. Notations. In the sequel, let Ω be a bounded open domain in the n-dimensional Euclidean space $\mathbf{R}^n$ of points $x = (x^1, \ldots, x^n)$, $n \geq 2$. The open ball in $\mathbf{R}^n$ with center x_0 and radius R will be denoted by $B_R(x_0)$, and $B_R = B_R(0)$. For any open set

[1] For a different class of elliptic systems, this fact was also pointed out in **[15]**.

$D \subset \mathbf{R}^n$, $L_q(D, \mathbf{R}^N)$ is the usual Lebesgue class of functions $u = (u^1(x), \dots, u^N(x))$ with finite norm

$$\|u\|_{q,D} = \left(\int_D |u|^q \, dx\right)^{1/q} \quad \text{if } 1 \leqslant q < \infty,$$
$$\|u\|_{\infty,D} = \operatorname*{ess\,sup}_{x \in D} |u(x)| \quad \text{if } q = \infty.$$

By $H_q^1(D, \mathbf{R}^N)$ we denote the Sobolev space of $\mathbf{R}^N$-valued functions $u \in L_q(D, \mathbf{R}^N)$ possessing a weak gradient $\nabla u \in L_q(D, \mathbf{R}^{nN})$, and $D_\alpha u^i$ stands for the partial derivative $\partial u^i / \partial x^\alpha$ ($\alpha = 1, \dots, n$; $i = 1, \dots, N$). $H_{q,c}^1(D, \mathbf{R}^N)$ consists of all functions $u \in H_q^1(D, \mathbf{R}^N)$ with compact support in D, while $\mathring{H}_q^1(D, \mathbf{R}^N)$ is the closure of $H_{q,c}^1(D, \mathbf{R}^N)$ in $H_q^1(D, \mathbf{R}^N)$. Finally, a function u belongs to $H_{q,\mathrm{loc}}^1(D, \mathbf{R}^N)$ if $u \in H_q^1(D', \mathbf{R}^N)$ for every bounded subdomain D' whose closure $\overline{D'}$ is contained in D ($D' \subset\subset D$).

If $u \in L_1(B_R(x_0), \mathbf{R}^N)$, we set

$$\fint_{B_R(x_0)} u \, dx = \frac{1}{\operatorname{meas} B_R(x_0)} \int_{B_R(x_0)} u \, dx.$$

Occasionally, this mean value is also abbreviated by $\bar{u}_R$. We shall not distinguish between a function $u \in L_1(D, \mathbf{R}^N)$ and its various representatives. For instance, $x_0 \in D$ is called a Lebesgue point of u if there exists a function $\tilde{u}$: $D \to \mathbf{R}^N$ coinciding with u almost everywhere on D and satisfying

$$\lim_{\rho \to 0} \fint_{B_\rho(x_0)} |\tilde{u}(x) - \tilde{u}(x_0)| \, dx = 0.$$

Note, in particular, that in the case $u \in L_\infty(D, \mathbf{R}^N)$ the condition

$$\lim_{\rho \to 0} \fint_{B_\rho(x_0)} |u(x) - \bar{u}_\rho|^2 \, dx = 0$$

holds for every Lebesgue point $x_0 \in D$ of u.

Throughout the paper, we use the Einstein summation convention: Repeated Latin indices $i, k, l, \dots$ are to be summed from 1 to N, repeated Greek indices $\alpha, \beta, \dots$ from 1 to n.

2. Example. The example presented here is a modification of a counterexample due to Frehse **[1]**. Among others, it will be shown that elliptic systems for which a Liouville theorem holds may at the same time possess discontinuous bounded weak solutions.

Let $n \geqslant 2$, and for $x = (x^1, \dots, x^n)$ set $r = ((x^1)^2 + (x^2)^2)^{1/2}$. On $\Omega = \{x \in \mathbf{R}^n : |x| < \exp(-2)\}$ we define the function

$$u(x) = \left(u^1(x), u^2(x)\right) = \left(\sin(\log|\log r|),\ \cos(\log|\log r|)\right), \tag{2.1}$$

which belongs to the class $H_2^1 \cap L_\infty(\Omega, \mathbf{R}^2)$ and which is a weak solution of the system

$$-\Delta u^1 = \frac{2(u^1 + u^2)}{1 + |u|^2} |\nabla u|^2, \quad -\Delta u^2 = \frac{2(u^2 - u^1)}{1 + |u|^2} |\nabla u|^2 \tag{2.2}$$

(cf. [**1**]). By adding and subtracting the equations in (2.2), we infer that

$$
(2.3)\qquad
\begin{aligned}
-\Delta(u^1+u^2) &= \frac{4u^2}{1+|u|^2}|\nabla u|^2 =: f^1(u,\nabla u),\\
-\Delta(u^2-u^1) &= \frac{-4u^1}{1+|u|^2}|\nabla u|^2 =: f^2(u,\nabla u).
\end{aligned}
$$

The system (2.3) can be rewritten in the form

$$(2.4)\qquad -D_\alpha\left(a_{ik}^{\alpha\beta}D_\beta u^k\right) = f^i(u,\nabla u) \qquad (i=1,2),$$

where

$$
(2.5)\qquad a_{ik}^{\alpha\beta} = \begin{cases} 0 & \text{if } \alpha\neq\beta,\\ 1 & \text{if } \alpha=\beta \text{ and } 1\leqslant i\leqslant k\leqslant 2,\\ -1 & \text{if } \alpha=\beta,\ i=2 \text{ and } k=1.\end{cases}
$$

An easy computation yields that $a_{ik}^{\alpha\beta}\xi_\alpha^i\xi_\beta^k = |\xi|^2$ for any $\xi\in\mathbf{R}^{2n}$. Moreover, the right-hand side $f=(f^1,f^2)$ in (2.3), (2.4) is a real analytic function of its arguments and satisfies the inequalities

$$(2.6)\qquad |f(u,\nabla u)|\leqslant 2|\nabla u|^2,$$

$$(2.7)\qquad u\cdot f(u,\nabla u)\leqslant 0.$$

The one-sided condition (2.7) implies that in the case $n=2$ a Liouville theorem holds for the system (2.4) (cf. Meier [**16**, Theorem 4], for an even more general result). On the other hand, we have seen that the function u defined by (2.1) is a discontinuous weak solution of (2.4) in Ω. Let us point out some other striking consequences of our example.

By multiplying u with a cut-off function $\eta\in C_c^\infty(\Omega)$, $\eta=1$ on $B_{1/9}(0)$, one obtains a discontinuous weak solution $v=(v^1,v^2)\in \mathring{H}_2^1\cap L_\infty(\Omega,\mathbf{R}^2)$ of the system

$$(2.8)\qquad -D_\alpha\left(a_{ik}^{\alpha\beta}D_\beta v^k\right) = f^i(v,\nabla v)+g^i(x) \qquad (i=1,2),$$

where $g^i\in L_\infty(\Omega)$ are suitably defined.

If $n=2$, there exists also a Hölder continuous weak solution of (2.8) with zero boundary values on $\partial\Omega$. This follows from a result due to Frehse [**3**], taking into account that $a_{ik}^{\alpha\beta}$ are constant coefficients and that the one-sided condition (2.7) holds. Surprisingly, the situation is quite different in the case of two-dimensional elliptic systems with *diagonal* principal part, where a one-sided condition implies the Hölder continuity of *all* bounded weak solutions in $\mathring{H}_2^1$ (cf. Wiegner [**19**]).

As a final remark, we mention that Giaquinta and Giusti [**5**] have proved a theorem of Meyers-Elcrat type (cf. [**18**]) for small solutions of systems with quadratic growth. The above example shows the impossiblity of replacing this smallness assumption by a one-sided condition (2.7) (see also §4). In fact, the solution u of (2.4) does not belong to the class $H^1_{2+\varepsilon,\mathrm{loc}}(\Omega,\mathbf{R}^2)$ for $\varepsilon>0$ ($n\geqslant 2$).

3. Local minima of quadratic multiple integrals. In this section, we are concerned with local minimum points of quadratic multiple integrals

$$(3.1)\qquad F(u,\Omega)=\int_\Omega a_{ik}^{\alpha\beta}(x,u)D_\alpha u^iD_\beta u^k\,dx,$$

where $a_{ik}^{\alpha\beta}(x, u)$ are bounded continuous functions on $\Omega \times \mathbf{R}^N$ satisfying the ellipticity condition

$$a_{ik}^{\alpha\beta}(x, u)\xi_\alpha^i\xi_\beta^k \geqslant \lambda|\xi|^2 \quad \text{for any } \xi \in \mathbf{R}^{nN} \text{ and some } \lambda > 0. \tag{3.2}$$

A function u is called a *local minimum* of F in Ω if $u \in H^1_{2,\mathrm{loc}}(\Omega, \mathbf{R}^N)$ and if

$$F(u, S(\phi)) \leqslant F(u + \phi, S(\phi))$$

holds for any function $\phi \in H^1_2(\Omega, \mathbf{R}^N)$ with compact support $S(\phi)$ in Ω. Moreover, let $x_0 \in \Omega$ and B_R be an arbitrary ball in $\mathbf{R}^n$. Then we define the integral

$$F_{x_0}(u, B_R) = \int_{B_R} a_{ik}^{\alpha\beta}(x_0, u)D_\alpha u^i D_\beta u^k\,dx. \tag{3.3}$$

A function $u_0 \in H^1_{2,\mathrm{loc}}(\mathbf{R}^n, \mathbf{R}^N)$ is called an *entire local minimum* of F_{x_0} if u_0 is a local minimum of F_{x_0} in every ball $B_R \subset \mathbf{R}^n$. Finally, we say that F has the *Liouville property* in $x_0 \in \Omega$ if any bounded entire local minimum of F_{x_0} is necessarily a constant vector.

The following theorems state that for bounded local minima of (3.1) the Liouville property is equivalent to a regularity result.

THEOREM 1. *Suppose that the coefficients $a_{ik}^{\alpha\beta}(x, u)$ satisfy the conditions above, and that F has the Liouville property in all points of Ω. If u is a bounded local minimum of F in Ω then $u \in C^\sigma(\Omega, \mathbf{R}^N)$ for any $\sigma < 1$. Moreover, the σ-Hölder norm of u on every interior subdomain $\Omega' \subset\subset \Omega$ can be estimated by an expression depending only on $n, N, a_{ik}^{\alpha\beta}, \sigma$, the essential maximum of $|u|$, and on the distance from Ω' to the boundary of Ω.*

THEOREM 2. *Suppose that F does not have the Liouville property in some point $x_0 \in \Omega$. Then there exists a bounded entire local minimum u_0 of F_{x_0} with the property that x_0 is not a Lebesgue point of u_0. In particular, u_0 is discontinuous in x_0.*

PROOF OF THEOREM 1. Let $\Omega' \subset\subset \Omega$ be fixed and set $\bar{R} = \frac{1}{4}\operatorname{dist}(\Omega', \partial\Omega)$. We shall show that for every $\delta > 0$ there exists a radius ρ_δ, $0 < \rho_\delta < \bar{R}$, depending only on $n, N, a_{ik}^{\alpha\beta}, \|u\|_{\infty,\Omega}$, and on δ, such that

$$\rho^{2-n}\int_{B_\rho(x)} |\nabla u|^2\,dx < \delta \tag{3.4}$$

holds for any $x \in \overline{\Omega'}$ and any $\rho \leqslant \rho_\delta$. The assertion then follows as in [**6**], in particular, since inequality (5.11) of [**6**] is also valid in the case when the coefficients $a_{ik}^{\alpha\beta}$ are not *uniformly* continuous and $u \in L_\infty(\Omega, \mathbf{R}^N)$. In order to verify that the C^σ-estimate obtained in this way does not depend upon the Dirichlet integral of u, one may use the inequality

$$\begin{aligned}\int_{B_{\bar{R}}(x)} |\nabla u|^2\,dx &\leqslant C_0\left(n, N, \frac{\Lambda}{\lambda}\right)\bar{R}^{-2}\int_{B_{2\bar{R}}(x)} |u - \bar{u}_{2\bar{R}}|^2\,dx \\ &\leqslant \bar{C}_0\bar{R}^{n-2}\|u\|^2_{\infty,\Omega}\end{aligned} \tag{3.5}$$

for $x \in \overline{\Omega'}$ (see [**6**, estimate (4.4)]). Here, $\Lambda > 0$ is an upper bound for $|a_{ik}^{\alpha\beta}|$, and $\bar{u}_{2\bar{R}}$ denotes the mean value of u on $B_{2\bar{R}}(x)$.

We prove (3.4) by contradiction, assuming that for some $\delta > 0$ there exist points $x_\nu \in \overline{\Omega'}$, radii $\rho_\nu < \bar{R}$, $\rho_\nu \to 0$ as $\nu \to \infty$, and local minima $u^{(\nu)}$ of F in Ω whose L_∞-norms on Ω are uniformly bounded by a constant $M > 0$ such that

$$\rho_\nu^{2-n} \int_{B_{\rho_\nu}(x_\nu)} |\nabla u^{(\nu)}|^2 \, dx \geqslant \delta > 0 \quad \text{for all } \nu \in \{1, 2, \dots\}. \tag{3.6}$$

Without loss of generality we can also assume that $x_\nu \to x_0 = 0 \in \overline{\Omega'}$ as $\nu \to \infty$. Then the blown-up functions

$$u_\nu(x) = u^{(\nu)}(x_\nu + \rho_\nu x) \tag{3.7}$$

are local minima of the integrals

$$F^\nu(u_\nu) = \int_{B_{2R_\nu}(0)} a_{ik}^{\alpha\beta}(x_\nu + \rho_\nu x, u_\nu) D_\alpha u_\nu^i D_\beta u_\nu^k \, dx \tag{3.8}$$

in $B_{2R_\nu} = B_{2R_\nu}(0)$, where $R_\nu = \bar{R} \cdot \rho_\nu^{-1}$.

Now let $R > 0$ be fixed and let $\nu_0 = \nu_0(R)$ be determined in such a way that $R_\nu \geqslant R$ for $\nu \geqslant \nu_0$. By assumption, we have $|u_\nu| \leqslant M$ on B_{2R}, whence an estimate similar to (3.5) yields the uniform boundedness of the Dirichlet integrals $\int_{B_R} |\nabla u_\nu|^2 \, dx$ ($\nu \geqslant \nu_0$). Thus, there exists a subsequence of $\{u_\nu\}$ (again denoted by $\{u_\nu\}$) such that $\{u_\nu\}_{\nu \geqslant \nu_0(R)}$ converges weakly in $H_2^1(B_R, \mathbf{R}^N)$, strongly in $L_2(B_R, \mathbf{R}^N)$, and pointwise almost everywhere on B_R for any $R > 0$. Obviously, the limit function u_0 belongs to the class $H_{2,\mathrm{loc}}^1 \cap L_\infty(\mathbf{R}^n, \mathbf{R}^N)$. In order to show that u_0 is an entire local minimum of F_0, we first note that for fixed $R > 0$ the coefficients $a_{ik}^{\alpha\beta}(x_\nu + \rho_\nu x, w)$, $\nu \geqslant \nu_0(R)$, converge to $a_{ik}^{\alpha\beta}(0, w)$ as $\nu \to \infty$ (uniformly on bounded subsets of $B_R \times \mathbf{R}^N$), by the continuity of $a_{ik}^{\alpha\beta}(x, w)$.

From the first part of the proof of Lemma 1 in [7], which rests upon uniform (local) $H_{2+\varepsilon}^1$-bounds for the minima u_ν combined with a weak lower semicontinuity result, we thus infer that u_0 is a bounded local minimum of F_0 in $B_R(0)$ for any $R > 0$.

The Liouville property now implies that u_0 is a constant vector. Moreover, since $u^{(\nu)}$ minimizes F locally in $B_{2\bar{R}}(x_\nu)$, we obtain

$$\begin{aligned} \rho_\nu^{2-n} \int_{B_{\rho_\nu}(x_\nu)} |\nabla u^{(\nu)}|^2 \, dx &\leqslant C_0\left(n, N, \frac{\Lambda}{\lambda}\right) \rho_\nu^{-n} \int_{B_{2\rho_\nu}(x_\nu)} |u^{(\nu)} - u_0|^2 \, dx \\ &= C_0 \int_{B_2(0)} |u_\nu - u_0|^2 \, dx \end{aligned} \tag{3.9}$$

(see also (3.5)), where the second integral has been transformed according to (3.7). As ν tends to infinity, the right-hand side of (3.9) converges to zero, whence we arrive at a contradiction to our assumption (3.6). This concludes the proof of Theorem 1.

Proof of Theorem 2. By hypothesis, there exists a bounded entire local minimum $u \in H_{2,\mathrm{loc}}^1(\mathbf{R}^n, \mathbf{R}^N)$ of F_{x_0} which is not a constant vector. Let M be an upper bound for $|u|$ on $\mathbf{R}^n$, and assume without loss of generality that $x_0 = 0$. We shall construct a singular minimum u_0 by a blow-down procedure. For this purpose, we consider the functions

$$(3.10)\qquad u_\nu(x) = u(\nu x), \qquad \nu = 1, 2, \ldots,$$

which are uniformly bounded entire local minima of $F_{x_0} = F_0$. The same reasoning as above yields that a subsequence of u_ν converges on every ball $B_R = B_R(0)$ weakly in H_2^1, strongly in L_2, and pointwise almost everywhere to a bounded entire local minimum u_0 of F_0. Denoting the mean value of u_0 on $B_{2\rho}$ by $\bar{u}_{0,2\rho}$ and taking the estimate (3.5) into account, we thus obtain

$$(3.11)\qquad \begin{aligned} \rho^{-n}\int_{B_{2\rho}} |u_0 - \bar{u}_{0,2\rho}|^2\,dx &\geqslant \liminf_{\nu\to\infty} (\nu\rho)^{-n}\int_{B_{2\nu\rho}} |u - \bar{u}_{0,2\rho}|^2\,dx \\ &\geqslant \frac{1}{C_0}\liminf_{\nu\to\infty} (\nu\rho)^{2-n}\int_{B_{\nu\rho}} |\nabla u|^2\,dx \quad \text{for any } \rho > 0. \end{aligned}$$

From this the assertion of Theorem 2 follows once we have shown that

$$(3.12)\qquad \liminf_{R\to\infty} R^{2-n}\int_{B_R} |\nabla u|^2\,dx > 0.$$

For fixed $R > 0$, let $v \in H_2^1(B_R, \mathbf{R}^N)$ be the solution of the variational problem

$$\int_{B_R} a_{ik}^{\alpha\beta}(0, \bar{u}_{0,R}) D_\alpha v^i D_\beta v^k\,dx \to \min, \qquad v - u \in \mathring{H}_2^1(B_R, \mathbf{R}^N).$$

On account of the maximum principle (cf. [**5**, Lemma 1.3]),

$$(3.13)\qquad |v| \leqslant C_1 M \quad \text{on } B_R,$$

where $C_1 \geqslant 1$ is a constant depending only on n, N, and Λ/λ. Moreover, since the coefficients $a_{ik}^{\alpha\beta}(0, w)$ are bounded on $\mathbf{R}^N$ and uniformly continuous on $\{w \in \mathbf{R}^N\colon |w| \leqslant C_1 M\}$, there exists a bounded, continuous, increasing, concave function $\omega\colon \mathbf{R}^+ \to \mathbf{R}^+$ satisfying $\omega(0) = 0$ such that

$$(3.14)\qquad |a_{ik}^{\alpha\beta}(0, w_1) - a_{ik}^{\alpha\beta}(0, w_2)| \leqslant \omega\left(|w_1 - w_2|^2\right) \quad \text{if } |w_1|, |w_2| \leqslant C_1 M.$$

By virtue of (3.13), (3.14), the (slightly modified) calculations in [**6**, pp. 39–41] yield the estimate

$$(3.15)\qquad \int_{B_\rho} |\nabla u|^2\,dx \leqslant C_2\left(\left(\frac{\rho}{R}\right)^n + \left[\omega\left(C_3 R^{2-n}\int_{B_R}|\nabla u|^2\,dx\right)\right]^{\varepsilon/(2+\varepsilon)}\right)\int_{B_R}|\nabla u|^2\,dx$$

for any $\rho < R$ with positive constants C_2, C_3, and ε independent of ρ and R.

Now let $\tau = 1/2C_2\ (< 1)$ and choose $\delta > 0$ in such a way that

$$(3.16)\qquad \left[\omega(C_3\delta)\right]^{\varepsilon/(2+\varepsilon)} \leqslant \tau^n.$$

Since u is not constant on $\mathbf{R}^n$, the function $\Phi(\rho) = \rho^{2-n}\int_{B_\rho}|\nabla u|^2\,dx$ takes a positive value $\Phi(R_1)$ for some $R_1 > 0$. If $R \geqslant R_1$ and $\Phi(R) \leqslant \delta$, we infer from (3.15) that

$$\begin{aligned} \Phi(\tau R) &\leqslant C_2\left\{\tau^2 + \left[\omega(C_3\Phi(R))\right]^{\varepsilon/(2+\varepsilon)}\tau^{2-n}\right\}\Phi(R) \\ &\leqslant 2C_2\tau^2\Phi(R) \leqslant \tau\Phi(R) \leqslant \Phi(R), \end{aligned}$$

whence by iteration $\Phi(\tau^m R) \leqslant \Phi(R)$ $(m = 1, 2, \ldots)$. One easily sees that in this case $\Phi(R) \geqslant \tau^{n-2}\Phi(R_1)$. Therefore, the estimate

$$\Phi(R) \geqslant \min\{\delta, \tau^{n-2}\Phi(R_1)\} > 0 \tag{3.17}$$

holds for any $R \geqslant R_1$. In particular, (3.12) follows, and Theorem 2 is completely proved.

4. Quasilinear elliptic systems. Let us consider second-order quasilinear elliptic systems of the form

$$-D_\alpha\left(a_{ik}^{\alpha\beta}(x, u) D_\beta u^k\right) = b_{ikl}^{\alpha\beta}(x, u) D_\alpha u^k D_\beta u^l + g_i(x) \qquad (i = 1, \ldots, N), \tag{4.1}$$

where $a_{ik}^{\alpha\beta}$ and $b_{ikl}^{\alpha\beta}$ are bounded continuous coefficients on $\Omega \times \mathbf{R}^N$ satisfying

$$\begin{gathered} a_{ik}^{\alpha\beta}(x, u)\xi_\alpha^i \xi_\beta^k \geqslant \lambda|\xi|^2, \quad |a_{ik}^{\alpha\beta}(x, u)| \leqslant \Lambda, \\ \left(\sum_{i=1}^N \left[\sum b_{ikl}^{\alpha\beta}(x, u)\xi_\alpha^k \xi_\beta^l\right]^2\right)^{1/2} \leqslant \mu|\xi|^2 \end{gathered} \tag{4.2}$$

for every $\xi \in \mathbf{R}^{nN}$ with positive constants λ, Λ and μ. Moreover, suppose that

$$g = (g_1, \ldots, g_N) \in L_p(\Omega, \mathbf{R}^N) \quad \text{for some } p > n/2, \ \|g\|_{p,\Omega} \leqslant G. \tag{4.3}$$

A function u is called a *bounded weak solution* of (4.1) in Ω if $u \in H_2^1 \cap L_\infty(\Omega, \mathbf{R}^N)$ and if

$$\int_\Omega \left\{ a_{ik}^{\alpha\beta}(x, u) D_\beta u^k D_\alpha \phi^i - b_{ikl}^{\alpha\beta}(x, u) D_\alpha u^k D_\beta u^l \phi^i - g_i(x)\phi^i \right\} dx = 0 \tag{4.4}$$

holds for any test vector $\phi \in H_{2,c}^1 \cap L_\infty(\Omega, \mathbf{R}^N)$. Moreover, we say that u_0 is a *bounded entire weak solution* of the system

$$-D_\alpha\left(a_{ik}^{\alpha\beta}(x_0, u_0) D_\beta u_0^k\right) = b_{ikl}^{\alpha\beta}(x_0, u_0) D_\alpha u_0^k D_\beta u_0^l \qquad (i = 1, \ldots, N), \tag{4.1$_{x_0}$}$$

$x_0 \in \Omega$, if $u_0 \in H_{2,\mathrm{loc}}^1 \cap L_\infty(\mathbf{R}^n, \mathbf{R}^N)$ and if u_0 is a weak solution of $(4.1)_{x_0}$ in every ball $B_R \subset \mathbf{R}^n$.

DEFINITION. Let $M > 0$ and $\varepsilon > 0$ be arbitrary.

(i) The system (4.1) has the *M-restricted Liouville property* in $x_0 \in \Omega$ if and only if any bounded entire weak solution u_0 of $(4.1)_{x_0}$ with $\|u_0\|_{\infty,\mathbf{R}^n} \leqslant M$ is necessarily a constant vector.

(ii) The system (4.1) is said to have the *(M, ε)-restricted Liouville property* in $x_0 \in \Omega$ if any bounded entire weak solution u_0 of $(4.1)_{x_0}$ satisfying $\|u_0\|_{\infty,\mathbf{R}^n} \leqslant M$ and, in addition,

$$\left(⨍_{B_R(x_0)} |\nabla u_0|^{2+\varepsilon}\, dx\right)^{1/(2+\varepsilon)} \leqslant \gamma_1 \left(⨍_{B_{2R}(x_0)} |\nabla u_0|^2\, dx\right)^{1/2} \leqslant \gamma_2 R^{-1} \tag{4.5}$$

for sufficiently large radii $R > 0$ (with positive constants γ_1, γ_2 independent of R) has to be a constant vector.

Obviously, the M-restricted Liouville property implies the (M, ε)-restricted Liouville property. We may now state our main results.

THEOREM 3. *Suppose that the coefficients* $a_{ik}^{\alpha\beta}$ *and* $b_{ikl}^{\alpha\beta}$ *of the system* (4.1) *are bounded and continuous, and that* (4.2), (4.3) *hold. Let* $u \in H^1_{2+\varepsilon,\mathrm{loc}}(\Omega, \mathbf{R}^N)$, $2 < 2 + \varepsilon \leqslant 2p$, *be a bounded weak solution of* (4.1) *in* Ω *satisfying* $|u| \leqslant M$ *and*

$$\left(\fint_{B_{2\rho}} |\nabla u|^2\,dx\right)^{1/2} \leqslant K_1\rho^{-1}, \tag{4.6}$$

$$\left(\fint_{B_\rho} |\nabla u|^{2+\varepsilon}\,dx\right)^{1/(2+\varepsilon)} \leqslant K_2\left\{\left(\fint_{B_{2\rho}} |\nabla u|^2\,dx\right)^{1/2} + \left(\fint_{B_{2\rho}} |g|^{(2+\varepsilon)/2}\,dx\right)^{1/(2+\varepsilon)}\right\} \tag{4.7}$$

for any ball $B_{4\rho} = B_{4\rho}(x) \subset \Omega$ *with* $\rho \leqslant \rho_0 \leqslant 1$.

If the system (4.1) *has the* (M, ε)*-restricted Liouville property in in all pointts of* Ω *then* $u \in C^\sigma(\Omega, \mathbf{R}^N)$ *for any* $\sigma < \min(1, 2 - n/p)$. *Moreover, the* σ*-Hölder norm of* u *on every interior subdomain* $\Omega' \subset\subset \Omega$ *can be estimated in terms of* n, N, $a_{ik}^{\alpha\beta}$, $b_{ikl}^{\alpha\beta}$, σ, p, G, M, ε, K_1, K_2, ρ_0, *and the distance from* Ω' *to the boundary of* Ω.

COROLLARY 1. *Assume that the conditions* (4.2), (4.3) *hold and that* u *is a bounded weak solution of* (4.1) *in* Ω *satisfying* $|u| \leqslant M$ *with*

$$2\mu M < \lambda. \tag{4.8}$$

If the system (4.1) *has the* M*-restricted Liouville property in all points of* Ω *then* $u \in C^\sigma(\Omega, \mathbf{R}^N)$ *for any* $\sigma < \min(1, 2 - n/p)$. *Moreover, one obtains local bounds for the Hölder norms of* u *depending on* u *and* g *only via the quantities* M, p *and* G.

THEOREM 4. *Assume that in some point* $x_0 \in \Omega$ *the system* (4.1) *does not have the* (M, ε)*-restricted Liouville property. Then there exists a bounded entire weak solution* u_0 *of* $(4.1)_{x_0}$ *with the properties that* $|u_0| \leqslant M$ *on* $\mathbf{R}^n$ *and that* x_0 *is not a Lebesgue point of* u_0. *In particular,* u_0 *is discontinuous in* x_0.

COROLLARY 2. *Suppose that the system* $(4.1)_{x_0}$ *possesses a bounded entire weak solution* u *which is not constant on* $\mathbf{R}^n$. *If* $|u| \leqslant M$ *and*

$$2\mu M < \lambda, \tag{4.8}$$

then there also exists a bounded entire weak solution u_0 *of* $(4.1)_{x_0}$, $|u_0| \leqslant M$, *for which* x_0 *is not a Lebesgue point. Moreover,* u_0 *can be obtained from* u *by a blow-down procedure.*

REMARKS. (a) The estimate (4.6) is satisfied, for instance, if the *one-sided condition*

$$u^i b_{ikl}^{\alpha\beta}(x, u) D_\alpha u^k D_\beta u^l \leqslant \theta a_{ik}^{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^k \tag{4.9}$$

holds with some constant $\theta < 1$. In this case, K_1 depends only on n, N, λ, Λ, θ, M, p, and G, as can be verified by inserting the test vector $\phi = \eta^2 u$ into (4.4), where $\eta \in C_c^\infty(B_{4\rho})$, $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ on $B_{2\rho}$, $|\nabla\eta| \leqslant 1/\rho$.

(b) Roughly speaking, Theorem 3 states that the (restricted) Liouville property implies Hölder continuity of those bounded weak solutions u of (4.1) whose gradients satisfy a reverse Hölder inequality (4.7). It should be noted that also the partial regularity results for (4.1) due to Giaquinta and Giusti (cf. [**4**, **5**]) rest entirely upon the estimate (4.7). This exhibits the connection between Liouville's property, almost everywhere and everywhere regularity.

(c) The assertion of Corollary 2 does not hold if (4.8) is replaced by the one-sided condition

$$u^i b_{ikl}^{\alpha\beta}(x,u)\xi_\alpha^k \xi_\beta^l \leqslant 0 \quad \text{for every } \xi \in \mathbf{R}^{nN}.$$

Under this weaker assumption, the blow-down argument due to Ivert [**12**] only shows that it is impossible to obtain uniform bounds for the modulus of continuity of solutions of $(4.1)_{x_0}$, while a discontinuous solution can, in general, not be constructed by this method. Indeed, by Example 3.4 in [**16**], there exists a nonconstant, bounded entire C^2-solution u of a system (4.1) with coefficients $a_{ik}^{\alpha\beta} = \delta^{\alpha\beta}\delta_{ik}$ and $b_{ikl}^{\alpha\beta}(x,u) = \delta^{\alpha\beta}\delta_{kl}d_i(x,u)$, where $d_i(x,u)$ are bounded continuous functions satisfying the estimate $u^i d_i(x,u) \leqslant 0$. Straightforward additional computations yield that the functions d_i may also be redefined in such a way that they do not depend upon the variable x. Thus, one arrives at a system of the form $(4.1)_{x_0}$ for u. Since $\nabla u \in L_2 \cap L_{2+\varepsilon}(\mathbf{R}^n, \mathbf{R}^{nN})$ for any $\varepsilon > 0$, it follows, in particular, that the sequence of solutions $u_R(x) = u(Rx)$ converges to a constant vector as R tends to infinity, whence a discontinuous solution cannot be obtained from u by means of a blow-down procedure. Moreover, the one-sided condition and the specific structure of the coefficients $a_{ik}^{\alpha\beta}$ imply that if $\tilde{u}$ is any bounded weak solution of the system considered above in some open domain $\Omega \subset \mathbf{R}^n$, then all points of Ω are Lebesgue points of $\tilde{u}$ (cf. [**10**, Theorems 2.3 and 3.1]). Of course, this remark also shows the impossibility of extending Theorem 4 to elliptic systems for which the M-restricted Liouville property does not hold.

PROOF OF THEOREM 3. Let $\Omega' \subset\subset \Omega$ be fixed and set $\overline{R} = \frac{1}{4}\operatorname{dist}(\Omega', \partial\Omega)$. As in the proof of Theorem 1 we shall demonstrate that for every $\delta > 0$ there exists a radius $\rho_\delta \leqslant \min(\overline{R}, \rho_0)$ (depending only on n, N, $a_{ik}^{\alpha\beta}$, $b_{ikl}^{\alpha\beta}$, p, G, M, ε, K_1, K_2, ρ_0, $\overline{R}$, and on δ) such that

$$\rho^{2-n}\int_{B_\rho(x)} |\nabla u|^2\,dx < \delta \tag{4.10}$$

holds for any $x \in \overline{\Omega'}$ and any $\rho \leqslant \rho_\delta$. The assertion of Theorem 3 then follows from the partial regularity results in [**4, 5**], which are stated only for $p = \infty$ but which can easily be extended also to the case $p > n/2$.

For simplicity of notations, let us write (4.1) in the form

$$-D(a(x,u)Du) = b(x,u)Du\,Du + g.$$

We prove (4.10) by contradiction, assuming that for some $\delta > 0$ there exist points $x_\nu \in \overline{\Omega'}$, radii $\rho_\nu \leqslant \min(\overline{\mathrm{R}}, \rho_0)$, $\rho_\nu \to 0$ as $\nu \to \infty$, functions $g^{(\nu)} \in L_p(\Omega, \mathbf{R}^N)$ with $\|g^{(\nu)}\|_{p,\Omega} \leqslant G$, and bounded weak solutions $u^{(\nu)}$, $\|u^{(\nu)}\|_{\infty,\Omega} \leqslant M$, of

$$-D\big(a(x,u^{(\nu)})Du^{(\nu)}\big) = b(x,u^{(\nu)})\,Du^{(\nu)}\,Du^{(\nu)} + g^{(\nu)} \quad \text{in } \Omega \tag{4.11}$$

satisfying the estimates (4.6) and (4.7) with u and g replaced by $u^{(\nu)}$ and $g^{(\nu)}$, respectively, such that

$$\rho_\nu^{2-n}\int_{B_{\rho_\nu}(x_\nu)} |\nabla u^{(\nu)}|^2\,dx \geqslant \delta > 0 \quad \text{for all } \nu \in \{1,2,\dots\}. \tag{4.12}$$

Without loss of generality we can also assume that $x_\nu \to x_0 = 0 \in \overline{\Omega'}$ as $\nu \to \infty$.

Our proof is now organized as follows. In Step 1 it will be shown that a suitable subsequence of the blown-up functions $u_\nu(x) = u^{(\nu)}(x_\nu + \rho_\nu x)$ converges weakly in $H^1_{2+\varepsilon,\text{loc}}$ to a bounded entire weak solution u_0, $|u_0| \leqslant M$, of the system

$$-D(a(0, u_0)\, Du_0) = f_0, \tag{4.13}$$

the right-hand side f_0 of which belongs to the class $L_{1+\varepsilon/2,\text{loc}}(\mathbf{R}^n, \mathbf{R}^N)$. The main part of the proof consists in verifying that

$$f_0 = b(0, u_0)\, Du_0\, Du_0. \tag{4.14}$$

This is done in Step 3, after we have established the remarkable result that the weakly convergent subsequence of $\{u_\nu\}$ actually converges *strongly* in $H^1_{2,\text{loc}}$ (Step 2). The improved convergence is used once more in Step 4 in order to derive the estimate (4.5). By virtue of the (M, ε)-restricted Liouville property we finally arrive at a contradiction to our assumption (4.12).

Step 1. For $|x| < R_\nu := \min(\bar{R}, \rho_0)\rho_\nu^{-1}$ set

$$\begin{aligned} u_\nu(x) &= u^{(\nu)}(x_\nu + \rho_\nu x), \qquad & a_\nu(x) &= a(x_\nu + \rho_\nu x, u_\nu(x)), \\ b_\nu(x) &= b(x_\nu + \rho_\nu x, u_\nu(x)), & g_\nu(x) &= \rho_\nu^2 g^{(\nu)}(x_\nu + \rho_\nu x). \end{aligned} \tag{4.15}$$

By transforming (4.11) it is easily seen that u_ν is a bounded weak solution of the system

$$-D(a_\nu(x)\, Du_\nu) = b_\nu(x)\, Du_\nu\, Du_\nu + g_\nu(x) =: f_\nu \tag{4.16}$$

in $B_{R_\nu} = B_{R_\nu}(0)$ and satisfies

$$|u_\nu| \leqslant M \quad \text{in } B_{R_\nu}. \tag{4.17}$$

Let $R > 0$ be fixed and $\nu_0 = \nu_0(R)$ be determined in such a way that $R_\nu \geqslant R$ for any $\nu \geqslant \nu_0$. In the following, suppose that $\nu \geqslant \nu_0$. Since

$$\left(\fint_{B_R} |\nabla u_\nu|^q\, dx \right)^{1/q} = \left(\fint_{B_{R\rho_\nu}(x_\nu)} |\nabla u^{(\nu)}|^q\, dx \right)^{1/q}$$

and

$$\left(\fint_{B_{2R\rho_\nu}(x_\nu)} |g^{(\nu)}|^{(2+\varepsilon)/2}\, dx \right)^{2/(2+\varepsilon)} \leqslant \left(\fint_{R_{2R\rho_\nu}(x_\nu)} |g^{(\nu)}|^p\, dx \right)^{1/p},$$

we infer by virtue of (4.7), (4.3) that

$$\begin{aligned} &\left(\fint_{B_R} |\nabla u_\nu|^{2+\varepsilon}\, dx \right)^{1/(2+\varepsilon)} \\ &\qquad \leqslant K_2 \left\{ \rho_\nu \left(\fint_{B_{2R\rho_\nu}(x_\nu)} |\nabla u^{(\nu)}|^2\, dx \right)^{1/2} + \gamma \rho_\nu^{1-n/2p} R^{-n/2p} G^{1/2} \right\}, \end{aligned} \tag{4.18}$$

where γ is a constant depending only on n and p. Finally, by combining (4.18) and (4.6) (with u replaced by $u^{(\nu)}$), one obtains the estimate

$$\begin{aligned} \left(\fint_{B_R} |\nabla u_\nu|^{2+\varepsilon}\, dx \right)^{1/(2+\varepsilon)} &\leqslant K_2 \left\{ \left(\fint_{B_{2R}} |\nabla u_\nu|^2\, dx \right)^{1/2} + \gamma \rho_\nu^{1-n/2p} R^{-n/2p} G^{1/2} \right\} \\ &\leqslant K_2 \left\{ K_1 R^{-1} + \gamma \rho_\nu^{1-n/2p} R^{-n/2p} G^{1/2} \right\} \end{aligned} \tag{4.19}$$

for any $\nu \geqslant \nu_0(R)$. Note that $n/2p < 1$, whence the right-hand side of (4.19) is uniformly bounded. Together with (4.17) this yields uniform $H^1_{2+\varepsilon}$-bounds for the functions $u_\nu, \nu \geqslant \nu_0$, on B_R. Thus, there exists a subsequence of $\{u_\nu\}$ (again denoted by $\{u_\nu\}$) with the property that for any $R > 0$,

(4.20) $\{u_\nu\}_{\nu \geqslant \nu_0(R)}$ converges to some function $u_0 \in H^1_{2+\varepsilon,\mathrm{loc}}(\mathbf{R}^n, \mathbf{R}^N)$ weakly in $H^1_{2+\varepsilon}(B_R, \mathbf{R}^N)$, strongly in $L_2(B_R, \mathbf{R}^N)$, and pointwise almost everywhere on B_R.

Obviously,

$$|u_0| \leqslant M \quad \text{on } \mathbf{R}^n. \tag{4.21}$$

Moreover, we have

$$\left(\int_{B_R} |g_\nu|^p \, dx\right)^{1/p} = \rho_\nu^{2-n/p}\left(\int_{B_{R\rho_\nu}(x_\nu)} |g^{(\nu)}|^p \, dx\right)^{1/p} \leqslant \rho_\nu^{2-n/p} G$$

for the inhomogeneous terms g_ν, $\nu \geqslant \nu_0(R)$, in (4.15), (4.16). Since $p \geqslant 1 + \varepsilon/2$ and $2 - n/p > 0$, this implies, in particular, that

(4.22) $\{g_\nu\}_{\nu \geqslant \nu_0(R)}$ converges to zero strongly in $L_{1+\varepsilon/2}(B_R, \mathbf{R}^N)$.

Finally, the nonlinearities f_ν in (4.16) are uniformly bounded in $L_{1+\varepsilon/2}(B_R, \mathbf{R}^N)$ for $\nu \geqslant \nu_0(R)$, on account of the boundedness of the coefficients $b^{\alpha\beta}_{ikl}$. Hence one may select a further subsequence such that for any $R > 0$,

(4.23) $\{f_\nu\}_{\nu \geqslant \nu_0(R)}$ converges weakly in $L_{1+\varepsilon/2}(B_R, \mathbf{R}^N)$ to some function $f_0 \in L_{1+\varepsilon/2,\mathrm{loc}}(\mathbf{R}^n, \mathbf{R}^N)$.

Now it is not difficult to show that u_0 is a bounded entire weak solution of the system (4.13). In fact, let $R > 0$ and $\phi \in C_c^\infty(B_R, \mathbf{R}^N)$ be arbitrary, and set $a_0(x) = a(0, u_0(x))$. By (4.16) we have

$$\begin{aligned} 0 &= \int_{B_R} \{a_\nu(x)\, Du_\nu D\phi - f_\nu \cdot \phi\}\, dx \\ &= \int_{B_R} \{a_0(x) Du_\nu D\phi - f_\nu\phi + [a_\nu(x) - a_0(x)]\, Du_\nu D\phi\}\, dx \end{aligned} \tag{4.24}$$

for $\nu \geqslant \nu_0$. The boundedness and continuity of the coefficients $a^{\alpha\beta}_{ik}$ and (4.20) imply that $|a_\nu(x) - a_0(x)|$ is uniformly bounded on B_R and converges to zero pointwise almost everywhere. Letting ν tend to infinity, and taking (4.20), (4.23), and the dominated convergence theorem into account, one arrives at

$$\int_{B_R} \{a_0(x)\, Du_0 D\phi - f_0\phi\}\, dx = 0. \tag{4.25}$$

Obviously, (4.25) also holds for any $\phi \in H^1_{2,c} \cap L_\infty(B_R, \mathbf{R}^N)$.

Step 2. We shall demonstrate that u_ν converges to u_0 *strongly* in H^1_2 on every ball B_R. Fix $R > 0$ and choose $\eta \in C_c^\infty(B_R, \mathbf{R}^N)$ satisfying $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ on $B_{R/2}$. Since the functions $|f_\nu|$ and $|\nabla u_\nu| |\nabla(u_\nu\eta)|$ are uniformly bounded in $L_{1+\varepsilon/2}(B_R)$, we infer by virtue of Hölder's inequality that

(4.26)

$$\lim_{\nu \to \infty} \int_{B_R} f_\nu \cdot (u_\nu - u_0)\eta \, dx = \lim_{\nu \to \infty} \int_{B_R} [a_\nu(x) - a_0(x)]\, Du_\nu D(u_\nu\eta)\, dx = 0,$$

where we have also used the uniform boundedness and the convergence properties of $|u_\nu(x) - u_0(x)|$ and $|a_\nu(x) - a_0(x)|$. Furthermore, $D_\beta u_\nu^k$ tends to $D_\beta u_0^k$ weakly in $L_2(B_R)$, whence

$$(4.27) \qquad \lim_{\nu\to\infty} \int_{B_R} a_{ik}^{\alpha\beta}(0, u_0) D_\beta u_\nu^k \, u_\nu^u D_\alpha \eta \, dx = \int_{B_R} a_{ik}^{\alpha\beta}(0, u_0) D_\beta u_0^k \, u_0^i D_\alpha \eta \, dx.$$

Now we take $\phi = u_0\eta$ and $\phi = u_\nu\eta$ as test vectors in (4.25) and (4.24), respectively. From (4.23), (4.26) it then follows that

$$(4.28) \qquad \int_{B_R} a_0(x)\, Du_0 D(u_0\eta)\, dx = \int_{B_R} f_0 \cdot u_0\eta\, dx = \lim_{\nu\to\infty} \int_{B_R} f_\nu \cdot u_0\eta\, dx$$
$$= \lim_{\nu\to\infty} \int_{B_R} f_\nu \cdot u_\nu\eta\, dx = \lim_{\nu\to\infty} \int_{B_R} a_\nu(x)\, Du_\nu D(u_\nu\eta)\, dx$$
$$= \lim_{\nu\to\infty} \int_{B_R} a_0(x) Du_\nu D(u_\nu\eta)\, dx.$$

By combining (4.28) and (4.27) one gets

$$(4.29) \qquad \lim_{\nu\to\infty} \int_{B_R} a_0(x) Du_\nu Du_\nu\, \eta\, dx = \int_{B_R} a_0(x)\, Du_0\, Du_0\, \eta\, dx$$

and, therefore,

$$(4.30) \qquad \lim_{\nu\to\infty} \int_{B_R} a_0(x)[Du_\nu - Du_0]\,[Du_\nu - Du_0]\eta\, dx = 0,$$

using once more the fact that ∇u_ν tends to ∇u_0 weakly in $L_2(B_R, \mathbf{R}^{nN})$. Finally, the properties of η and the ellipticity condition in (4.2) imply that

$$(4.31) \qquad \lim_{\nu\to\infty} \int_{B_{R/2}} |\nabla u_\nu - \nabla u_0|^2\, dx = 0,$$

where $R > 0$ is arbitrary.

Step 3. From the strong convergence of ∇u_ν it now follows easily that the function f_0 in (4.23), (4.25) coincides with $b(0, u_0)\, Du_0\, Du_0$. In fact, for any $R > 0$ we have

$$(4.32) \qquad \int_{B_R} |b_\nu(x)\, Du_\nu\, Du_\nu - b(0, u_0)\, Du_0\, Du_0|\, dx$$
$$\leqslant \int_{B_R} |\{b_\nu(x) - b(0, u_0)\}\, Du_\nu\, Du_\nu|\, dx$$
$$+ \int_{B_R} |b(0, u_0)\{Du_\nu\, Du_\nu - Du_0\, Du_0\}|\, dx.$$

As $\nu \to \infty$, the first integral on the right-hand side of (4.32) tends to zero, because of the boundedness and continuity of the coefficients $b_{ikl}^{\alpha\beta}$ and the uniform boundedness of $|\nabla u_\nu|^2$ in $L_{1+\varepsilon/2}(B_R)$.

Moreover, (4.31) yields that the second integral approaches zero, whence by (4.22) the sequence $\{f_\nu\}_{\nu \geqslant \nu_0(R)}$ converges to $b(0, u_0)\, Du_0\, Du_0$ strongly in $L_1(B_R, \mathbf{R}^N)$ for any $R > 0$. From (4.23) and (4.25) one thus obtains that u_0 is a bounded entire weak solution of the system

$$-D\big(a(0, u_0)\, Du_0\big) = b(0, u_0)\, Du_0\, Du_0. \tag{4.33}$$

Step 4. Finally, we let ν tend to infinity in (4.19). Since the first integral is lower semicontinuous with respect to weak convergence in $H^1_{2+\varepsilon}(B_R, \mathbf{R}^N)$ and since $⨍_{B_{2R}} |\nabla u_\nu|^2\, dx$ tends to $⨍_{B_{2R}} |\nabla u_0|^2\, dx$, we arrive at

$$\left(⨍_{B_R} |\nabla u_0|^{2+\varepsilon}\, dx \right)^{1/(2+\varepsilon)} \leqslant K_2 \left(⨍_{B_{2R}} |\nabla u_0|^2\, dx \right)^{1/2} \leqslant K_1 K_2 R^{-1}. \tag{4.34}$$

By virtue of the (M, ε)-restriced Liouville property, u_0 is thus a constant vector, whence

$$\rho_\nu^{2-n} \int_{B_{\rho_\nu}(x_\nu)} |\nabla u^{(\nu)}|^2\, dx = \int_{B_1(0)} |\nabla u_\nu|^2\, dx \to 0 \quad \text{as } \nu \to \infty. \tag{4.35}$$

This contradicts our assumption (4.12), and (4.10) follows. Theorem 3 is completely proved.

PROOF OF COROLLARY 1. Under the hypotheses of Corollary 1, the estimate (4.7) holds for $B_{4\rho}(x) \subset \Omega$, $\rho \leqslant \rho_0 = 1$, with suitable constants K_2, $\varepsilon > 0$ depending on u and g only via the quantities M and p. This can be seen from the proof of Theorem 2.3 in [**4**, p. 144]. Moreover, (4.8) implies that the one-sided condition (4.9) is satisfied with $\theta = 1/2$. Taking Remark (a) into account and using the fact that the system (4.1) has the M-restricted and hence also the (M, ε)-restricted Liouville property in all points of Ω, one can apply Theorem 3, and the assertion of Corollary 1 follows.

PROOF OF THEOREM 4. Let u be a bounded entire weak solution of $(4.1)_{x_0}$ which is not constant and satisfies

$$|u| \leqslant M \quad \text{on } \mathbf{R}^n, \tag{4.36}$$

$$\left(⨍_{B_R(x_0)} |\nabla u|^{2+\varepsilon}\, dx \right)^{1/(2+\varepsilon)} \leqslant \gamma_1 \left(⨍_{B_{2R}(x_0)} |\nabla u|^2\, dx \right)^{1/2} \leqslant \gamma_2 R^{-1} \tag{4.37}$$

for $R \geqslant R_2$. We may assume without loss of generality that $x_0 = 0$. By a slight modification of the calculations presented in [**4**, pp. 181–183], one concludes from (4.37) that

$$\int_{B_\rho} |\nabla u|^2\, dx \leqslant C \left\{ \left(\frac{\rho}{R} \right)^n + \chi \left(R^{2-n} \int_{B_R} |\nabla u|^2\, dx \right) \right\} \int_{B_R} |\nabla u|^2\, dx \tag{4.38}$$

for all ρ, R with $2R_2 < \rho < R$. Here, $\chi(t)$ is a nonnegative function tending to zero as t tends to zero.

The last estimate corresponds to (3.15) in the proof of Theorem 2, and a similar reasoning as there yields that

$$L := \liminf_{R \to \infty} R^{2-n} \int_{B_R} |\nabla u|^2\, dx > 0. \tag{4.39}$$

Now consider the sequence

$$u_\nu(x) = u(\nu \cdot x) \qquad (\nu = 1, 2, \ldots) \tag{4.40}$$

of entire solutions of $(4.1)_0$ satisfying $|u_\nu| \leqslant M$ on $\mathbf{R}^n$. By virtue of (4.36) and (4.37), the $H^1_{2+\varepsilon}$-norms of u_ν are uniformly bounded on every ball B_R, whence one can select a subsequence of $\{u_\nu\}$ converging to some function $u_0 \in H^1_{2+\varepsilon,\mathrm{loc}}(\mathbf{R}^n, \mathbf{R}^N)$ pointwise almost everywhere and weakly in $H^1_{2+\varepsilon}(B_R, \mathbf{R}^N)$ for any $R > 0$. As in Steps 2 and 3 of the proof of Theorem 3 it follows that this subsequence converges *strongly* in $H^1_2(B_R, \mathbf{R}^N)$ for every $R > 0$ and that u_0 is a bounded entire weak solution of $(4.1)_0$ with $\|u_0\|_{\infty,\mathbf{R}^n} \leqslant M$. Hence, (4.39) yields

$$\begin{aligned} \rho^{2-n}\int_{B_\rho} |\nabla u_0|^2\,dx &\geqslant \liminf_{\nu\to\infty} \rho^{2-n}\int_{B_\rho} |\nabla u_\nu|^2\,dx \\ &= \liminf_{\nu\to\infty} (\nu\rho)^{2-n}\int_{B_{\nu\rho}} |\nabla u|^2\,dx \geqslant L > 0 \end{aligned} \tag{4.41}$$

for any $\rho > 0$.

In order to show that $x_0 = 0$ is not a Lebesgue point of u_0, we fix an arbitrary $\rho > 0$ and apply (4.37) with $R = \nu\rho$, $\nu > R_2/\rho$. Rewriting the estimate in terms of u_ν and passing to the limit $\nu \to \infty$, we infer that

$$\left(\fint_{B_\rho} |\nabla u_0|^{2+\varepsilon}\,dx\right)^{1/(2+\varepsilon)} \leqslant \gamma_1\left(\fint_{B_{2\rho}} |\nabla u_0|^2\,dx\right)^{1/2} \leqslant \gamma_2\rho^{-1}. \tag{4.42}$$

By Hölder's inequality it may be assumed that $\varepsilon \leqslant 2$.

Now choose $\eta \in C^\infty_c(B_\rho)$ in such a way that $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ on $B_{\rho/2}$, $|\nabla\eta| \leqslant 4/\rho$. Moreover, let $\bar{u}_{0,\rho} = \fint_{B_\rho} u_0\,dx$ and take $\phi = \eta^2(u_0 - \bar{u}_{0,\rho})$ as test vector for $(4.1)_0$. This yields the estimate

$$\int_{B_{\rho/2}} |\nabla u_0|^2\,dx \leqslant \frac{c_1}{\rho^2}\int_{B_\rho} |u_0 - \bar{u}_{0,\rho}|^2\,dx + c_2\int_{B_\rho} |u_0 - \bar{u}_{0,\rho}|\,|\nabla u_0|^2\,dx, \tag{4.43}$$

where c_1 and c_2 are constants independent of ρ. The last integral is estimated by means of Hölder's inequality together with (4.42). Since $\varepsilon \leqslant 2$ and $|u_0 - \bar{u}_{0,\rho}| \leqslant 2M$, we thus obtain

(4.44)

$$\left(\frac{\rho}{2}\right)^{2-n}\int_{B_{\rho/2}} |\nabla u_0|^2\,dx \leqslant c_3\left\{\left(\fint_{B_\rho} |u_0 - \bar{u}_{0,\rho}|^2\,dx\right)^{\varepsilon/(2+\varepsilon)} + \fint_{B_\rho} |u_0 - \bar{u}_{0,\rho}|^2\,dx\right\}.$$

Combining (4.41) and (4.44) and letting ρ tend to zero, one finally arrives at

$$\liminf_{\rho\to 0} \int_{B_\rho} |u_0 - \bar{u}_{0,\rho}|^2\,dx > 0. \tag{4.45}$$

This completes the proof of Theorem 4.

PROOF OF COROLLARY 2. As in the proof of Corollary 1 we conclude that u satisfies the estimate (4.5) for any $R > 0$ with suitable constants ε, γ_1, $\gamma_2 > 0$. The assertion then follows from Theorem 4.

REMARKS. (i) The results of this section can be generalized to systems of the form

$$(4.1)' \qquad -D_\alpha\big(a_{ik}^{\alpha\beta}(x,u)D_\beta u^k\big)=\mathring{f}_i(x,u,\nabla u)+g_i(x) \quad (i=1,\dots,N),$$

$$(4.1)'_{x_0} \quad -D_\alpha\big(a_{ik}^{\alpha\beta}(x_0,u_0)D_\beta u_0^k\big)=\mathring{f}_i(x_0,u_0,\nabla u_0) \qquad (i=1,\dots,N),$$

where $\mathring{f}=(\mathring{f}_1,\dots,\mathring{f}_N)$ is a continuous function on $\Omega\times\mathbf{R}^N\times\mathbf{R}^{nN}$ satisfying the conditions

$$|\mathring{f}(x,u,\xi)|\leqslant\mu|\xi|^2,\quad \mathring{f}(x,u,t\xi)=t^2\mathring{f}(x,u,\xi)$$

for all $(x,u,\xi)\in\Omega\times\mathbf{R}^N\times\mathbf{R}^{nN}$ and any $t\geqslant 0$.

In order to carry over Step 3 of the proof of Theorem 1, one may, for instance, use the fact that

$$\mathring{f}(x_\nu+\rho_\nu x,u_\nu(x),\nabla u_\nu(x)) = \mathring{f}\left(x_\nu+\rho_\nu x,u_\nu(x),\frac{\nabla u_\nu(x)}{1+|\nabla u_\nu(x)|}\right)\{1+|\nabla u_\nu(x)|\}^2,$$

and that a subsequence of $\{\nabla u_\nu\}_{\nu\geqslant\nu_0(R)}$ converges to ∇u_0 pointwise almost everywhere on B_R.

The remaining arguments are essentially the same as before.

(ii) By a modification of our method, one obtains similar connections between the regularity in a neighborhood of the boundary of bounded solutions (resp. minima) with prescribed smooth Dirichlet data and a "boundary Liouville property" concerning bounded solutions (resp. minima) in the half-space $\mathbf{R}^n_+=\{x\in\mathbf{R}^n\colon x^n>0\}$ whose boundary values on $\partial\mathbf{R}^n_+$ are constant. In this case, a partial regularity result up to the boundary must be proved, and the blow-up and blow-down procedure is then performed on half-balls instead of balls.

We note that a variant of this approach was used in **[13]** in order to establish a boundary regularity theorem for the minima of a special class of variational integrals (3.1). Furthermore, we should like to refer to a recent paper by Giaquinta, Nečas, John and Stará **[9]** treating boundary value problems and related boundary Liouville properties for certain elliptic systems of a structure different from (4.1).

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